OCR
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A Level Further Mathematics B (MEI)
Y434 Numerical Methods
Sample Question Paper

Date – Morning/Afternoon
Time allowed: 1 hour 15 minutes

OCR supplied materials:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)
- Scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.**
- Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- In each question you must show sufficient detail of the method(s) which you are using.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **8** pages.

1 **(i)** Solve the following simultaneous equations.

$$x + \quad y = 1$$
$$x + 0.99y = 2$$

[2]

(ii) The coefficient 0.99 is correct to two decimal places. All other coefficients in the equations are exact. With the aid of suitable calculations, explain why your answer to part **(i)** is unreliable. **[3]**

i. $x + y = 1 \Rightarrow x = 1 - y$

$x + 0.99y = 2 \quad x = 2 - 0.99y$

$1 - y = 2 - 0.99y$

$-1 = 0.01y$

$\underline{y = -100}$

$x = 1 - (-100)$

$\underline{x = 101}$

ii. $y = \frac{-1}{1 - 0.99} \Rightarrow \frac{-1}{1 - 0.995} < y < \frac{-1}{1 - 0.985}$

$-200 < y < -66.67$

$x = 1 - y \Rightarrow 1 - (-66.67) < x < 1 - (-200)$

$67.67 < x < 201$

because we are subtracting 2 numbers that are very similar, the range for x & y is huge

2 The following spreadsheet printout shows the bisection method being applied to the equation $f(x)=0$, where $f(x)=e^x-x^2-2$. Some values of $f(x)$ are shown in columns B and D.

	A	B	C	D	E	F	G
1	a	f(a)	b	f(b)	$(a+b)/2$	f(($a+b$)/2)	mpe
2	1	-0.28172	2	1.389056	1.5	0.231689	0.5
3	1	-0.28172	1.5	0.231689	1.25	-0.072157	0.25
4	1.25	-0.07216	1.5	0.231689	1.375	0.064452	0.125
5	1.25	-0.07216	1.375	0.064452	1.3125	-0.007206	0.0625
6	1.3125	-0.00721	1.375	0.064452	1.34375	0.027728	0.03125

(i) The formula in cell A3 is $\boxed{\text{=IF(F2>0, A2, E2)}}$. State the purpose of this formula. **[1]**

(ii) The formula in cell C3 is $\boxed{\text{=IF(F2>0, ..., ...)}}$. What are the missing cell references? **[1]**

(iii) In which row is the magnitude of the maximum possible error (mpe) less than 5×10^{-7} for the first time? **[2]**

i. check if 1 needs to replace 1.5 by checking the sign of F2

ii. =IF(F2>0, E2, C2)

iii. $0.5^{n-1} < 5\times10^{-7}$

$0.5^{n} < 2.5\times10^{-7}$

$n=20: \ 0.5^{20} = 9.5\times10^{-7}$

$n=21: \ 0.5^{21} = 4.77\times10^{-7}$

$n=22: \ 0.5^{22} = 2.38\times10^{-7}$

$2.38\times10^{-7} < 2.5\times10^{-7}$

so this happens on the <u>22nd row</u>

3 The equation $\sinh x + x^2 - 1 = 0$ has a root, α, such that $0 < \alpha < 1$.

(i) Verify that the iteration $x_{r+1} = \frac{1-\sinh x_r}{x_r}$ with $x_0 = 1$ fails to converge to this root. **[2]**

(ii) Use the relaxed iteration $x_{r+1} = (1-\lambda)x_r + \lambda\left(\frac{1-\sinh x_r}{x_r}\right)$ with $\lambda = \frac{1}{4}$ and $x_0 = 1$ to find α correct to 6 decimal places. **[2]**

i. $x_{r+1} = \frac{1-\sinh x_r}{x_r}$

$x_0 = 1$

$x_1 = -0.175$

$x_2 = -6.713\ldots$

$x_3 = -61.443$

so this does not converge

ii. $x_{r+1} = (1-\lambda)x_r + \lambda\left(\frac{1-\sinh x_r}{x_r}\right)$

$\lambda = \frac{1}{4}$

$x_0 = 1$

$x_1 = 0.706120$

$x_2 = 0.612353$

$x_3 = 0.601606$

$x_4 = 0.601403$

$x_5 = 0.601402$

$x_6 = 0.601402$

so $\alpha = 0.601402$

4 The table below gives values of a function $y = f(x)$.

x	0.2	0.3	0.35	0.4	0.45	0.5	0.6
$f(x)$	0.789 922	0.754 628	0.749 199	0.749 997	0.756 257	0.767 523	0.804 299

(i) Calculate three estimates of $\frac{dy}{dx}$ at $x = 0.4$ using the central difference method. **[4]**

(ii) State the value of $\frac{dy}{dx}$ at $x = 0.4$ to an appropriate degree of accuracy. Justify your answer. **[2]**

i. $h = 0.2$: $\frac{dy}{dx} \simeq \frac{f(0.6) - f(0.2)}{0.6 - 0.2}$

$= \frac{0.804299 - 0.789922}{0.4}$

$= 0.0359425$ ✓

$h = 0.1$: $\frac{dy}{dx} \simeq \frac{f(0.5) - f(0.3)}{0.5 - 0.3}$

$= \frac{0.767523 - 0.754628}{0.2}$

$= 0.064475$ ✓

$h = 0.05$: $\frac{dy}{dx} \simeq \frac{f(0.45) - f(0.35)}{0.45 - 0.35}$

$= \frac{0.756257 - 0.749199}{0.1}$

$= 0.07058$

ii. 0.07 is reasonable

$h = 0.05$ is the smallest h, so gives the most accurate estimate

5 A vehicle is moving in a straight line. Its velocity at different times is recorded and shown below. The velocities are recorded to 5 significant figures and the times may be assumed to be exact.

Time (t seconds)	5	10	12	15
Velocity (v metres per second)	5.1250	11.000	14.000	18.375

It is suggested initially that a quadratic model may be appropriate for this situation.

(i) Given that the vehicle is modelled as a particle with constant mass, what assumption about the net force acting on the vehicle leads to a quadratic model? **[1]**

(ii) Find Newton's interpolating polynomial of degree 2 to model this situation. Write your answer in the form $v = at^2 + bt + c$. **[4]**

(iii) Comment on whether this model appears to be appropriate. **[2]**

(iv) Use this model to find an approximation to the distance travelled over the interval $5 \le t \le 15$. **[2]**

Further investigation suggests that a cubic model may be more appropriate.

(v) What technique would you use to fit a cubic model to the data in the table? **[1]**

i. force is a linear function of time

ii. use the values of t = 5, 10, 15

	1st difference	2nd difference
5.1250		
	5.875	
11		1.5
	7.375	
18.375		

$$v = 5.125 + \frac{(t-5)}{5} \times (5.875) + \frac{(t-5)(t-10)}{5^2 \times 2!} \times 1.5$$

$$= 5.125 + 1.175t - 5.875 + \frac{3(t^2 - 15t + 50)}{100}$$

$$= 1.175t - 0.75 + 0.03t^2 - 0.45t + 1.5$$

$$v = 0.03t^2 + 0.725t + 0.75$$

iii. $f(12) = 0.03(12)^2 + 0.725(12) + 0.75$
$= 13.77$

13.77 is close to 14.000, so the model seems to be appropiate

iv. $S = \int v\,dt = \int_5^{15} (0.03t^2 + 0.725t + 0.75)\,dt$

$$= \left[0.01t^3 + 0.3625t^2 + 0.75t\right]_5^{15}$$

$$= 0.01(15)^3 + 0.3625(15)^2 + 0.75(15) - 0.01(5)^3 - 0.3625(5) - 0.75(5)$$

$$= 112.5$$

v. Lagrange's form of the interpolating polynomial

6 The secant method is to be used to solve the equation $x - \ln(\cos x) - 1 = 0$.

(i) Show that starting with $x_0 = -1$ and $x_1 = 0$ leads to the method failing to find the root between $x = 0$ and $x = 1$. **[3]**

The spreadsheet printout shows the application of the secant method starting with $x_0 = 0$ and $x_1 = 1$. Successive approximations to the root are in column E.

	A	B	C	D	E
1	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$	x_{n+2}
2	0	-1	1	0.6156265	0.6189549
3	1	0.6156265	0.6189549	-0.175846	0.7036139
4	0.6189549	-0.1758461	0.7036139	-0.025245	0.7178053
5	0.7036139	-0.0252451	0.7178053	0.0011619	0.7171808
6	0.7178053	0.0011619	0.7171808	-7.4E-06	0.7171848
7	0.7171808	-7.402E-06	0.7171848	-2.16E-09	0.7171848
8	0.7171848	-2.16E-09	0.7171848	3.997E-15	0.7171848

(ii) What feature of column B shows that this application of the secant method has been successful? **[1]**

(iii) Write down a suitable spreadsheet formula to obtain the value in cell E2. **[2]**

Some analysis of convergence is carried out, and the following spreadsheet output is obtained.

	A	B	C	D	E	F	G	H
1	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$	x_{n+2}			
2	0	-1	1	0.6156265	0.6189549	0.084659	0.167629	1.980053
3	1	0.6156265	0.6189549	-0.175846	0.7036139	0.0141913	-0.044	-3.10054
4	0.6189549	-0.1758461	0.7036139	-0.025245	0.7178053	-0.0006244	-0.00633	10.13727
5	0.7036139	-0.0252451	0.7178053	0.0011619	0.7171808	3.953E-06	0.000292	73.83899
6	0.7178053	0.0011619	0.7171808	-7.4E-06	0.7171848	1.154E-09	-1.8E-06	
7	0.7171808	-7.402E-06	0.7171848	-2.16E-09	0.7171848	-2.109E-15		
8	0.7171848	-2.16E-09	0.7171848	3.997E-15	0.7171848			

The formula in cell F2 is $\boxed{\text{=E3 – E2}}$. The formula in cell G2 is $\boxed{\text{=F3/F2}}$. The formula in cell H2 is $\boxed{\text{=F3/(F2\^{}2)}}$.

(iv) (*A*) Explain the purpose of each of these three formulae. **[3]**

(*B*) Explain the significance of the values in columns G and H in terms of the rate of convergence of the secant method. **[2]**

(v) Explain why the values in cells F6 and F7 are not 0. **[1]**

i. $f(x) = x - \ln\cos x - 1$

$$x_0 = -1,\ x_1 = 0,\ x_2 = 0 - \frac{(0-0-1)(0-(-1))}{(0-0-1)-(-1-\ln 0.5403)}$$

$$= 2.601636$$

to find x_3 we need $\ln(\cos(2\cdot 60\ldots))$

$\cos(2\cdot 60\ldots) = -0\cdot 8577$

$\ln(-0\cdot 8577)$ is undefined, so this method fails

ii. the values are converging to zero

iii. (A2*D2 − C2*B2)/(D2−B2)

iv. A) =E2−E1 finds the difference between the first 2 estimates

=F3/F2 finds the ratio of differences for the first two estimates

=F3/(F2^2) finds the ratio of differences of each estimate to the square of the previous

B) the values in column G suggest convergence is faster than 1st order.

the values in H suggest the convergence is slower than 2nd order

V. the values in column E are stored to greater accuracy than displayed

7 Fig. 7 shows the graph of $y = \mathrm{f}(x)$ for values of x from 0 to 1.

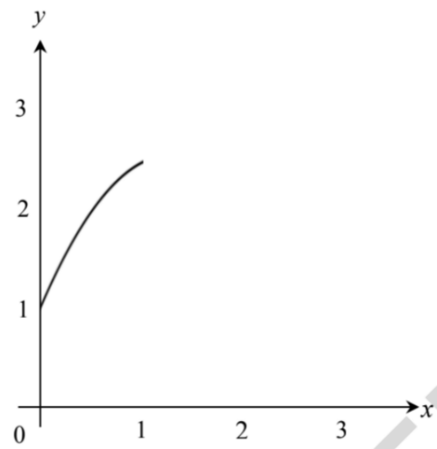

Fig. 7

The following spreadsheet printout shows estimates of $\int_0^1 \mathrm{f}(x)\mathrm{d}x$ found using the midpoint and trapezium rules for different values of h, the strip width.

	A	B	C
1	h	Midpoint	Trapezium
2	1	1.99851742	1.751283839
3	0.5	1.9638591	1.874900631
4	0.25	1.95135259	1.919379864
5	0.125	1.94682102	1.935366229

(i) Without doing any further calculation, write down the smallest possible interval which contains the value of the integral. Justify your answer. **[2]**

(ii) (*A*)
- Calculate the ratio of differences, r, for the sequence of estimates calculated using the trapezium rule.
- Hence suggest a value for r correct to 2 significant figures.
- Comment on your suggested value for r. **[3]**

(*B*)
- Use extrapolation to find an improved approximation to the value of the integral.
- State the value of the integral to two decimal places.
- Explain why this precision is secure. **[4]**

i. $1{\cdot}9353662285 \leq I < 1{\cdot}946821025$

the curve is concave, so curves upwards, meaning the trapezium rule provides an underestimate & the midpoint rule provides an overestimate

ii. A)

	differences	ratios
1.751283839		
	0.123616792	
1.874900631		0.3598155
	0.044479233	
1.919379864		0.3594119
	0.015986368	
1.935366229		

so $r \simeq 0.36$

the trapezium rule is a 2^{nd} order method, so r should be close to 0.25

B) $$1.93536623 + 0.01598637 \times \frac{0.36}{0.64}$$

$$= 1.9443586$$

So the value of the integral is 1.94 to 2 d.p., which agrees with T_8.

Using a similar approach with the sequence of estimates calculated using the midpoint rule, the approximation to the integral from extrapolation was found to be 1.944 27 correct to 5 decimal places.

(iii) Andrea uses the extrapolated midpoint rule value and the value found in part **(ii)** (B) to write down an interval which contains the value of the integral. Comment on the validity of Andrea's method. **[2]**

(iv) Use the values from the spreadsheet output to calculate an approximation to the integral using Simpson's rule with $h = 0.125$. Give your answer to 5 decimal places. **[2]**

Approximations to the integral using Simpson's rule are given in the spreadsheet output below. The number of applications of Simpson's rule is given in column N.

N	O	P	Q
n	Simpson	differences	ratio
1	1.91610623	0.01810005	0.3584931
2	1.93420628	0.00648874	0.3556525
4	1.94069502	0.00230774	0.3544828
8	1.94300275	0.00081805	0.3539885
16	1.94382081	0.00028958	0.3537638
32	1.94411039	0.00010244	0.3536568
64	1.94421283	3.623E-05	
128	1.94424906		

(v) Use the spreadsheet output to find the value of the integral as accurately as possible. Justify the precision quoted. **[6]**

END OF QUESTION PAPER

iii. extrapolation is only an approximate method, so Andrea's method is not appropriate

iv. When $h = 0{\cdot}125$, $I \simeq \dfrac{1{\cdot}91937986 + 2 \times 1{\cdot}95135259}{3}$

$= 1{\cdot}940695013$

$= \underline{1{\cdot}94070}$

v. extrapolation: $1.94424906 + 0.00003623 \times \frac{0.354}{1-0.354}$

$$= 1.944268914$$

so we can say that it is 1.9443 to 4 d.p.
because this agrees with 1.94424906